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Research Article

APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS IN NONLINEAR SECOND ORDER EQUATIONS

[1]**ADAPA DURGA MADHURI**
Associate Professor B. V. C Engineering College (A)
madhuriadapa2@gmail.com

[2]**P.V. NARESH SAGAR**
Asst. Professor B.V.C Engineering College(A)
sagar.pvn@gmail.com

[3]**PINISETTI GANGARAJU**
Asst. Professor B. V. C Engineering College(A)
laxmigangaraju@gmail.com

ABSTRACT:
In the present paper there is a detailed study for the abstract second order (in time) semi linear differential equation on Cauchy problem

$$u''(t) + Au'(t) + Bu(t) = f(t, u(t)),$$

in which A and B are (generally unbounded) operators which are linear in a Banach space. This sort of problems rise up frequently with in the study of PDE (partial differential equations). In regular we manage a nonlinear perturbation (likely related to spatial derivatives) through the linear terms, which incorporate higher order spatial derivatives. But opposite to the same old approach of lowering the hassle to a 1st order device in some "energy" norm space, we make the use of the factoring approach. The approach lets in the equation to be written as integral equation which consists a double integral related to the non linearity, reflecting the truth that the equation is 2nd order. There are numerous blessings to this technique with a purpose to be illustrated completely with in the examples. At first, we display that the equation is domestically posed well if the nonlinearity is going to satisfy the local Lipschitz condition

$$\| f(t, u_1) - f(t, u_2)\| \leqslant C_1 \|u_1 - u_2\|_A + C_2 \|u_1 - u_2\|_{B^{1/2}}$$

(on soaking up linear terms into f, higher order derivatives of the operators A and B want be retained.) For partial differential equations, this offers a great rule of thumb for figuring out if a positive hassle is locally properly posed. Energy strategies may be used to expose worldwide existence. Secondly, this technique applies to problems of each hyperbolic and parabolic.

EQUATION OF KLEIN-GORDON:
Here, the proof for the existence regarding a strong global solution and study of bifurcation of the heavily damped Klein-Gordon equation shall be done.

$$u_{tt} - 2\alpha\, \Delta u_t - \Delta u = \mu u - b\, |u|^{(q-1)}\, u, \qquad (3.1)$$

With $\mu \in R$, $\alpha > 0$, $q \geq 1$, in the domain which is bounded, $\Omega \subseteq R^n$, $n = 1,2,3$, of smooth 852 , the condition for boundary u (t, x) = 0 is $x \in \partial\Omega$. (With the spectral Decomposing instead of extending to the characteristic function, we can derive the local existence of $\Omega = R^3$. The argument for global existence and energy is the same.) If $\mu = 0$, $\alpha = 0$, and $1 \leq q \leq 3$, and there is a known weak global solution. the result shows that for $\alpha > 0$, $q > 3$ has a strong global solution, leading to the problems which are interesting: Set initial conditions, $q > 3$, representing the corresponding solution of equation (3.1) and The weak solution equation (3.1) corresponding to v, $\alpha = 0$.

lim $\alpha \to 0$, does it have a strong meaning? What's the connection between u α and v?

Lets put (3.1) in Complex Hilbert space L2(Ω), Δ represents closure of Laplacian relative to the L2 norm in the set

$$\{u \in C^2(\Omega) : u = 0 \text{ on } \partial\Omega\}$$

- Δ have these Eigenvalues, λ_j, j = 1, 2, ..., $\lambda_j > 0$ with $\lim_{j\to\infty} \lambda_j = \infty$ and the corresponding complete orthogonal eigenvector set.

Lets consider

$$A_k = \text{closure } (\alpha\Delta + (-1)^k(\alpha^2\Delta^2 + \Delta)^{1/2}) \text{ for } k = 1, 2$$

Then we see that A is the sum of operator which is non-positive self adjoint and an operator which is bounded (it is corresponded to eigenvalues which are finite, where

$$(\alpha^2\lambda_j^2 - \lambda_j) \text{ is negative}$$

Therefore semigroup T_k, k = 1, 2 which is generated by A_k.
Now let B_1 = I (identity) $B_2 = \Delta$.

$$\text{(i)} \quad \|f(u)\| \leqslant g(K),\ \|f(u) - f(v)\| \leqslant g(K)(\|\Delta(u-v)\| + \|u-v\|)$$

$$\text{(ii)} \quad \|\Delta u_{n+1}\| \leqslant C(\|\Delta u_0\| + \textstyle\int_0^t \|f_n(s)\|\, ds), \text{ and}$$

$$\text{(iii)} \quad (\|\Delta(u_{n+1} - u_n)\| \leqslant C \textstyle\int_0^t \|f_n(s) - f_{n-1}(s)\|\, ds$$

Where,

$$\|u\|,\ \|v\|,\ \|\Delta u\|,\ \|\Delta v\| \leqslant K,$$

g is as in (H1), C is a +ve constant which is independent of n and f, $f_n(s) \equiv f(u_n(s)) = \mu u_n - b\,|u_n|^{(q-1)}\,u_n$

Consider,

$$\|u\|_\infty = \sup_{x\in\Omega} |u(x)| \;\text{ and }\; \|u\|_p = \left(\int_\Omega |u|^p\,dx\right)^{1/p},\; p \geqslant 1$$

now, from the imbeddings of Sobolev,

$$\|u\|_\infty \leqslant C\,\|\Delta u\| \;\text{ and }\; \|u\|_p \leqslant C\,\|\Delta u\|$$

for C > 0. Therefore (i) is followed.

$$\|\Delta(u_{n+1}-u_n)\| = \left\|\Delta\int_0^t\int_s^t \exp((t-\tau)A_1)\exp((\tau-s)A_2)\,g(s)\,d\tau\,ds\right\|$$

$$= \left\|\sum_j (-\lambda_j)\int_0^t\int_s^t \exp((t-\tau)(-\alpha\lambda_j-\beta_j)) \times\exp((\tau-s)(-\alpha\lambda_j+\beta_j))\,g_j(s)\,d\tau\,ds\right\|$$

As (ii) and (iii) are similar, we shall verify only (iii),

from Operational Calculus for self adjoint operators, by using value of β_j .i.e.

$$(\alpha^2\lambda_j^2-\lambda_j)^{1/2} \text{ and } g_j(s) = \langle g(s), e_j\rangle$$

$$\|\Delta(u_{n+1}-u_n)\|^2 = \left\|\sum \frac{-\lambda_j}{2\beta_j}\int_0^t (\exp((t-s)(-\alpha\lambda_j+\beta_j)) - \exp((t-s)(-\alpha\lambda_j-\beta_j)))\,g_j(s)\,e_j\,ds\right\|^2$$

$$\leqslant C\left\|\int_0^t (T_2(t-s)-T_1(t-s))\,g(s)\,ds\right\|^2$$

here, $C = \sup_j |\lambda_j/(\alpha^2\lambda_j^2-\lambda_j)^{1/2}|$

We can separately consider a limited number of eigenvalues, where βj = 0 to get the same inequality).

$$\|\Delta(u_{n+1}(t)-u_n(t))\| \leqslant C\int_0^t \|g(s)\|\, ds.$$

Therefore(3.1) has a mild local solution.

Existence of a global solution which is real with b> 0 is followed by considering the following energies

$$E_1(t)=(\|\nabla u\|^2+\|u_t\|^2-\mu\,\|u\|^2)/2+b\,\|u^{(q+1)/2}\|^2/(q+1)$$

$$E_2(t)=[\alpha\,\|\Delta u\|^2+\|u_t\|^2+\|\nabla u\|^2-\mu\,\|u\|^2$$

$$+2b\,\|u^{(q+1)/2}\|^2/(q+1)]/2\alpha-\langle u_t,\Delta u\rangle$$

Where (.,.) represents inner product**s**. According to(3.1) and part of the integra**l** we **get**

$$\frac{dE_1(t)}{dt}=-2\alpha\,\|\nabla u_t\|^2$$

Itfo**llow**s that E_1 (t) **doe**s not increase**.I**f μ ≤ 0, there is no problem. We assume μ is a positive **number**. Now let us **ass**u**m**e $\|\mu\|^2$ is unbounded. Then there is an increasing sequence t_n, such that

$$\|u(t_n)\|^2>n,\quad n=1,2,...$$

$$B_n=\{x\in\Omega: |u(t_n)|^{q-1}>(q+1)(\mu+1)/2b\}$$

$$\int_{B_n} u^2(t_n)\,dx\to\infty \text{ as } n\to\infty$$

$$E_1(t_n)\geqslant\int_\Omega b\,\frac{|u(t_n)|^{(q+1)}}{(q+1)}-\frac{\mu}{2}\,|u(t_n)|^2\,dx$$

$$=\int_\Omega |u(t_n)|^2(2b\,|u(t_n)|^{q-1}/(q+1)-\mu)/2\,dx$$

$$\geqslant\int_{B_n} u^2(t_n)/2\,dx-C,$$

where C is some fixed positive value. Since $\int_{bn} U^2 (t_n)\,dx \to \infty$, $E_{1(t_n)} \to \infty$ it is a contradiction and therefore, $\|u\|$ is a bounded function in t. From the definition of E_1 we then see that $\|\nabla.u\|$, $\|u_t\|$ and $\|u^{(q+1)/2}\|$ bounded uniformly in t. We now consider E_2.

$$\frac{dE_2(t)}{dt}=\int_\Omega\left\{(\Delta u)\,\alpha\,\Delta u_t+u_t u_{tt}/\alpha+\nabla u_t\cdot\nabla u_t\right.$$
$$\left.-(\Delta u)\,u_{tt}+\nabla u\cdot\nabla u_t/\alpha+\frac{b}{\alpha}\,|u|^q\,u_t\,\text{sign}\,u+\mu u u_t/\alpha\right\}dt.$$

Since $u_{tt}=\alpha\Delta u_t+\Delta u-b|u|^{q-1}u+\mu u$, $\alpha\Delta u_t=u_{tt}-\Delta u+b|u|^{q-1}u-\mu u$, and $\nabla|\mathbf{u}|=\text{sign}\,u\nabla\mathbf{u}$ we get

$$\frac{dE_2(t)}{dt}=\int_\Omega-(\Delta u)^2+b\,|u|^q\,\text{sign}\,u\,\Delta u+\mu\,\nabla u\cdot\nabla u\,dx$$

$$=-\int_\Omega(\Delta u)^2+(b\,|u|^{q-1}-\mu)\,|\nabla u|^2\,dx.$$

since $\|\nabla\mathbf{u}\|$ is bounded ,

$$\frac{dE_2}{dt}(t)\leqslant C$$

for the constant $C \geq 0$. Thus $E_2(t) \leq Ct + C_1$, with C_1 a positive value(i.e. fixed). Using the inequality with

$$2\,|\langle\Delta u,u_t\rangle|\leqslant\beta\,\|\Delta u\|^2+\|u_t\|^2/\beta,$$

with β arbitrarily least and since $\|u\|$ and $\|u_t\|$ are bounded, here it is from E_2 that $\|\Delta u\|$ is bounded on certain intervals. Therefore, from Corollary 2.3 we have global existence of solutions of (3.1) for all $q \geq 1$. To show that solutions are strong solutions it is necessary to prove that u_{tt}, Δu_t, Δu are $L^2(\Omega)$. We show that Δu_t is, the others being similar. Using Lemma 2.1 on

$$u(t)=u_0(t)+\int_0^t\int_s^t T_1(t-\tau)\,T_2(\tau-s)\,f(u(s))\,d\tau\,ds$$

(and assuming the initial data are enough so that ,Δu_0, Δu_{0t} , Δu_{0tt} exist) we have after a substitution and a change in the order of integration that

$$\Delta u_t(t) = \Delta u_{0t} + \Delta \frac{d}{dt}\int_0^t T_2(t-\tau)\int_0^\tau T_1(\tau - s)\, f(u(s))\, ds\, d\tau$$

$$= u_{0t} + \Delta \int_0^t T_1(t-s)\, f(u(s))\, ds$$

$$+ A_2 \int_0^t T_2(t-\tau)\, \Delta \int_0^\tau T_1(\tau - s)\, f(u(s))\, ds\, d\tau.$$

Since (i) A_2 is bounded (its spectrum is bounded), (ii) A_1, A_2, Δ, T_1 and T_2 all reduced, (iii) $D(\Delta) = D(A_1)$ by the operational calculus of self- adjoint operators and

$$\text{(iv)} \quad \int_0^t T_j(t-s)\, f(s)\, ds \in D(A_j) \text{ if } f \in C(\mathbb{R}_+, \chi),\ j = 1, 2,$$

we have that considered terms is in $L^2(\Omega)$. We now study bifurcation by letting μ be a parameter and using the Center Manifold Theorem [4]. To do this we need to reduce (3.1) to a first order system (3.2)

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + F(u, v),$$

where $F(u, v) = (v, f(u))^T$,T is the transpose operator. It is easy to see that a strong solution u to (3.1) gives a strong solution to (3.2) with $u = u_t - A_1 u$.Therefore we have existence of global strong solutions of (3.2).

Can be mentioned that by the operational calculus for self adjoint operators and here, $2\alpha\lambda_1 \geq \alpha\lambda_j$ (large j), we have $D(\Delta^\gamma)=D(A^\gamma_1)$for $0 \leq \gamma \leq 1$.

Let $F((\phi, \psi)^T)= (\psi, \mu\phi\text{-}b|\phi|^{q-1})^T$. F have Frechet derivative F^1 from $D(A^\gamma_1) \oplus L^2(\Omega)$ into$L^2(\Omega)\oplus L^2(\Omega)$ which is continuous, and $F^1(0)= \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}$.

$$D(A_1^\gamma) \text{ is } \|\phi\|_\gamma^2 = \|\phi\|^2 + \|A_1^\gamma \phi\|^2.$$

Consider,

$$\mu = \lambda_1 + 2\alpha\lambda_1 \varepsilon$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 2\alpha\lambda_1 u \end{pmatrix} + G\begin{pmatrix} u \\ v \end{pmatrix}, \qquad (3.3)$$

$$\varepsilon' = 0,$$

$$A = \begin{pmatrix} A_1 & 1 \\ \lambda_1 & A_2 \end{pmatrix} \text{ and } G\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -b|u|^{q-1}u \end{pmatrix}.$$

$$(\phi, \psi)^{\mathrm{T}} = \sum (a_j e_j, b_j e_j)^{\mathrm{T}} = \sum (a_j, b_j)^{\mathrm{T}} e_j,$$

If,

then,

$$A\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum \begin{pmatrix} -\alpha\lambda_j - (\alpha^2\lambda_j^2 - \lambda_j)^{1/2} & 1 \\ \lambda_1 & -\alpha\lambda_j + (\alpha^2\lambda_j^2 - \lambda_j)^{1/2} \end{pmatrix}\begin{pmatrix} a_j \\ b_j \end{pmatrix} e_j$$

and,

$$\sigma(A) = \{\beta_j^{\pm} = -\alpha\lambda_j \pm (\alpha^2\lambda_j^2 - \lambda_j + \lambda_1)^{1/2}; j = 1, \ldots\}$$

Note that, $\beta^{+}_{1}=0$, and

$$\mathrm{Re}\{\beta_1^-, \beta_{j, j=2,\ldots}^{\pm}\} \leqslant -\delta < 0$$

and $\delta > 0.$

Eigenvectors associated with $\beta^{\pm}_1$ are

$$f^{\pm} = (1, \pm\alpha\lambda_1 + (\alpha^2\lambda_1^2 - \lambda_1)^{1/2})^{\mathrm{T}} e_1$$

Eigenvectors associated with $\beta^{\pm}_{1}$, j=2 are orthogonal to $f^{\pm}$

The projection along f^- onto f^+

$$P(a, b)^{\mathrm{T}} e_1 = \langle (\alpha\lambda_1 - (\alpha^2\lambda_1^2 - \lambda_1)^{1/2}, 1)^{\mathrm{T}}, (a, b)^{\mathrm{T}} \rangle f^+ / 2\alpha\lambda_1$$

The equation 3.3 can be rewritten as,

$$s'(t) f^+ = s(t)\varepsilon f^+ + P(0, -b\,|u|^q u)^{\mathrm{T}},$$

$$y'(t) = By(t) + (I - P)(0, -b\,|u|^q u)^{\mathrm{T}}, \qquad (3.4)$$

$$\varepsilon' = 0,$$

where y is in the space

$$H = \{f^-\} \oplus \{f^+, f^-\}^{\perp}.$$

with

$$\{f^+, f^-\}^{\perp}$$

The complement which is orthogonal of f+ and f- in L2(Ω), B could be a linear operator on H with $\mathrm{Re}\,\sigma(B) \leqslant -\delta.$. By Theorem 6.2.1, there's a Lipschit- zian native invariant manifold for some $\varepsilon_0 > 0,$ $S = \{\, y = h(s, \varepsilon);\, |s|, |\varepsilon| < \varepsilon_0 \}$ is also drawn by the normal equation. The flow in S can be represented by the following differential equation

$$s' f^+ = s\varepsilon f^+ + PG(sf^+ + h(s, \varepsilon))$$
$$\varepsilon' = 0.$$

From the corollary 6.2, we get, $\|h(s, \varepsilon)\| = O(s^2 + \varepsilon^2)$ as $(s, \varepsilon) \to (0, 0)$. Therefore,

$$PG(sf^+ + h(s, \varepsilon)) = -C\,|s|^{q-1}\, sf^+ + O(s^{q+1} + \varepsilon s^q)$$

As where, $(s, \varepsilon) \to (0, 0)$, where $C = b \int_{\Omega} |e_1|^{q-1}\, e_1^2\, dx/2\alpha\lambda_1 > 0.$ So the flow on the manifold S may be represented by the solution to

$$\begin{aligned} s' &= \varepsilon s - C\,|s|^{q-1}\, s + O(s^{q+1}, \varepsilon s^q) \\ \varepsilon' &= 0. \end{aligned} \tag{3.5}$$

If ε =0, the solution to (3.5) decays exponentially and thus the origin is asymptotically stable in Eq. (3.2). Likewise when ε = 0, the origin is stable. Suppose ε > 0 and small. Since (for ε small and fixed),

$$\varepsilon s - C\,|s|^{q-1}\, s + O(s^{q+1}, \varepsilon s^q) = 0 \tag{3.6}$$

has a negative and positive root (close to zero), the unstable manifold of the origin consists of two orbits which are stable connecting two points which are fixed to the origin.

We finally remark that analogous bifurcation results for (3.1) were derived by Webb when **Ω** = (0, π).

RESULT:

We have discussed the Cauchy problem for the abstract second order differential equation which is semi linear, where the linear operators in a Banach space. These problems often arise in the study of partial differential equations. As in usual we control a non linear perturbation by the linear terms, which contain higher order spatial derivatives. This method allows the equation to be written as an integral equation containing a double integral involving the nonlinearity, reflecting the fact that the equation is second order.

CONCLUSION:

The applications of partial differential equations in solving non linear second order differential equations is very interesting and helps us to solve the non linear second order differential equations. There have been many people working on this and this have been developed over years and may take advancements in the future coming years. The paper can be understood by keen observation and understanding of partial differential equations and non linear second order differential equations.

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