KURATOWSKI $\mathcal{I}$-CONVERGENT DOUBLE SEQUENCES OF A CLOSED SETS ON INTUITIONISTIC FUZZY NORMED SPACE

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Abstract

Motivated by the notion of Kuratowski convergence of sequences of closed set [20]. In this paper, we extend the concept of Kuratowski convergence to Kuratowski ideal convergence with respect to intuitionistic fuzzy normed space for a double sequence of closed sets and rectify some properties for this new defined double sequence spaces.

Keywords: Ideal, Filter, t-norm, t-conorm, double sequences, Intuitionistic fuzzy normed spaces.

1 Introduction

Theory of fuzzy sets was studied and introduced by Zadeh [21] in 1965. In past years, the fuzzy theory has emerged as the most active area of research in many branches of mathematics and engineering. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. Park [13] discussed the notion of intuitionistic fuzzy(IF-) metric spaces which is based both on the idea of IFS which was introduced by Atanassov [1] and the concept of a fuzzy metric space by George and Veeramani [6]. The notion of intuitionistic fuzzy norm space [15],certainly there are some situations where the ordinary norm does not work and the concept of intuitionistic fuzzy norm seems to be more suitable in such cases, that is, we can deal with such situations by modelling the inexactness of the norm in some situations.

The statistical convergence of a sequences came into existence in 1951 by Fast [4]. When difficulties comes in series summation then this new concept was introduced. In this new idea of convergence of a sequences was that the majority of elements converges and we do not care about what is going on with other elements. In 2006, M. Burgina and O. Duman [11] studied that the sequences come from real life sources, such as computation and measurement do not permit in a ordinary case, to test that they converge or statistically converge in the mathematical sense. Later on it was analysed by Friday [5] from the sequence point of view and linked it with the summability theory. $\mathcal{I}$-convergence is a generalization of the statistical convergence. It was studied at the initial stage by Kostyrko, Salat and Wilezynski [8]. Further it was studied by Salat [17]. Salat, Tripathy and Ziman [18], Demirci [3], Khan et.al [7] and many others [12, 16]. Kumar and Kumar [9] studied the concepts of $\mathcal{I}$-convergence and $\mathcal{I}^*$-

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convergence for sequences of fuzzy numbers.

In 1902, Painleve introduced the concepts of inner and outer limits for a sequence of sets in his lecture on analysis at the University of Paris; set convergence was defined as the equality of these two limits. This convergence has been popularized by Kuratowski in his famous book Topologie [10] and thus, often called Kuratowski convergence of sequences of sets.

2 Definitions and Prelimineries

We first recall the concepts of an ideal and a filter of sets :

Definition 2.1. *[8] If $\mathbb{N}\times\mathbb{N}$ be the set of Cartesian product of natural numbers, then a family of subsets $\mathcal{I}$ of $\mathbb{N}\times\mathbb{N}$ is called an ideal in $\mathbb{N}\times\mathbb{N}$ if*

(a) $\phi \in I$,

(b) $A, B \in I \Rightarrow A \cup B \in I$,

(c) For each $A \in I$ *and* $B \subseteq A$, *we have* $B \in I$.

Remark 2.1. *An ideal I is said to be non-trivial if $\mathcal{I} \neq 2^{\mathbb{N}\times\mathbb{N}}$.*

Definition 2.2. *[8] A non-empty set $\mathcal{F} \in 2^{\mathbb{N}\times\mathbb{N}}$ is said to be filter in $\mathbb{N}\times\mathbb{N}$ if*

(a) $\phi \notin \mathcal{F}$,

(b) For $A, B \in \mathcal{F}$, *we have* $A \cap B \in \mathcal{F}$,

(c) For each $A \in \mathcal{F}$ *with* $A \subseteq B$, *we have* $B \in \mathcal{F}$.

The following proposition expresses a relation between the notion of an ideal and filter :

Corollary 2.1. *For each ideal I, there is a filter $\mathcal{F}(\mathcal{I})$ associated with $\mathcal{I}$ defined as:*

$$\mathcal{F}(\mathcal{I}) = \{M \subseteq \mathbb{N}\times\mathbb{N} : \mathbb{N}\times\mathbb{N} - M \in \mathcal{I}\}.$$

In 2008, Das et al. [2] gave the notions of ideal convergence of double sequences in real line as well as in general metric spaces. They firstly investigate the porosity and category position of bounded ideal convergent double sequences.

Definition 2.3. *A nontrivial ideal I of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $k\times\mathbb{N}$ and $\mathbb{N}\times k$ belong to I for each $k \in \mathbb{N}$.*

It is easily see that a strongly admissible ideal is admissible also.

Let $I_0 = \{K \subset \mathbb{N}\times\mathbb{N} : (\exists m(K) \in \mathbb{N})(i, j \geq m(K) \Rightarrow (i, j) \notin K)\}$. Then I_0 is a nontrivial strongly admissible ideal and clearly an ideal I is strongly admissible if and only if $I_0 \subset I$.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}\times\mathbb{N}}$ is said to be property (AP), if every countable family of mutually disjoint sets $\{A_{ij}\}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{B_{ij}\}$ of sets such that each symmetric difference $A_{ij}\Delta B_{ij}$ is finite set for $(i, j) \in \mathbb{N}\times\mathbb{N}$ and $B = \bigcup_{ij=1} B_{ij} \in I$. Hence $B_{ij} \in I$ for each $(i, j) \in \mathbb{N}\times\mathbb{N}$.

Definition 2.4. *An element $\xi \in X$ is said to be $\mathcal{I}$-limit point of a sequence $x = (x_{ij})$ if there is a set $M = \{m_{11} < m_{12} < ... < m_{kl} < ...\} \subset \mathbb{N} \times \mathbb{N}$ such that $M \notin I$ and $\lim_{ij\to\infty} x_{ij} = \xi$. The set of all $\mathcal{I}$-limit points of a sequence x will be denoted by $\mathcal{I}(\bigwedge_x)$.*

Definition 2.5. *An element $\xi \in X$ is called $\mathcal{I}$-cluster point of a sequence $x = (x_{ij})$ if for each $\epsilon > 0$, we have a set $\{(i,j) \in \mathbb{N} : d(x_{ij}, \xi) < \epsilon\} \notin \mathcal{I}$. The set of all $\mathcal{I}$-cluster points of a sequence x will be denoted by $I(\Gamma_x)$.*

Let $\mathcal{L}_x$ denote the set of all limit points ξ of the double sequence x_{ij}; i.e., $\xi \in \mathcal{L}_\S$ if there exists an infinite set $K = \{k_{11} < k_{12} < ...\}$ such that $x_{kmn} \to \xi$ as $(m,n) \to \infty$.
Clearly, for an admissible ideal $\mathcal{I}$ we have $\mathcal{I}(\bigwedge_x) \subseteq \mathcal{I}(\Gamma_x) \subseteq \mathcal{L}_x$.

Lemma 2.1. *[18] K be a compact subset of X. Then we have $K \cap \mathcal{I}(\Gamma_x \neq \phi)$ for every $x = (x_{ij})$ with $\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \in K\} \notin \mathcal{I}$.*

The concept of $\mathcal{I}$-limit superior and inferior were studied and introduced by Demirci [3] as follows :
Let $\mathcal{I}$ be an admissible ideal and $x = (x_{ij})$ be a real number sequence.

$$\mathcal{I} - \limsup_{i,j\to\infty} x_{ij} := \begin{cases} \sup B_x & \text{if } B_x \neq \phi, \\ -\infty & \text{if } B_x = \phi, \end{cases}$$

$$\mathcal{I} - \liminf_{i,j\to\infty} x_{ij} := \begin{cases} \liminf A_x & \text{if } A_x \neq \phi, \\ \infty & \text{if } A_x = \phi, \end{cases}$$

where

$A_x := \{a \in \mathbb{R} : \{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} < a\} \notin \mathcal{I}\}$ and
$B_x := \{b \in \mathbb{R} : \{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} > b\} \notin \mathcal{I}\}$

Lemma 2.2. *[3] If $\beta = \mathcal{I} - \lim \sup_{i,j\to\infty} x_{ij}$ is finite, then for every $\epsilon > 0$,
$\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} > \beta - \epsilon\} \notin \mathcal{I}$ and $\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} > \beta + \epsilon\} \in \mathcal{I}$.
Conversely, if the above equations holds for every $\epsilon > 0$ then $\beta = \mathcal{I} - \lim \sup_{i,j\to\infty} x_{ij}$*

The dual statement for $I - \liminf$ is as follows :

Lemma 2.3. *[3] If $\alpha = \mathcal{I} - \lim \inf_{i,j\to\infty} x_{ij}$ is finite, then for every $\epsilon > 0$,
$\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} < \alpha + \epsilon\} \notin \mathcal{I}$ and $\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} < \alpha - \epsilon\} \in \mathcal{I}$.
Conversely, if the above equations holds for every $\epsilon > 0$ then $\alpha = \mathcal{I} - \lim \inf_{(i,j)\to\infty} x_{ij}$.*

Definition 2.6. *[7]A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:
(a) $*$ is associative and commutative,
(b) $*$ is continuous,
(c) $a * 1 = a$ for all $a \in [0,1]$,
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.*

Example 2.1. *Two typical examples of continuous t-norm are* $a * b = ab$ *and* $a * b = min(a, b)$.

Definition 2.7. *A binary operation* $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ *is said to be a continuous t-conorm if it satisfies the following conditions:*
(a) $\diamond$ *is associative and commutative,*
(b) $\diamond$ *is continuous,*
(c) $a \diamond 0 = a$ *for all* $a \in [0,1]$,
(d) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

Example 2.2. *Two typical examples of continuous t-conorm are* $a \diamond b = min(a + b, 1)$ *and* $a \diamond b = max(a, b)$.

We define the notion of intuitionistic fuzzy normed spaces with the help of continuous t-norms and continuous t- conorms as a generalization of fuzzy normed space due to Saadati and Vaezpour [14].

Definition 2.8. *[7]The five-tuple* $(X, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy normed space(for short, IFNS) if* X *is a vector space,* $*$ *is a continuous t-norm,* $\diamond$ *is a continuous t-conorm and* μ, ν *are fuzzy sets on* $X \times (0, \infty)$ *satisfying the following conditions for every* $x, y \in X$ *and* $s, t > 0$:
(a) $\mu(x,t) + \nu(x,t) \leq 1$,
(b) $\mu(x,t) > 0$,
(c) $\mu(x,t) = 1$ *if and only if* $x = 0$,
(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,
(e) $\mu(x,t) * \mu(y,s) \leq \mu(x + y, t + s)$,
(f) $\mu(x, .) : (0, \infty) \to [0,1]$ *is continuous,*
(g) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to 0} \mu(x,t) = 0$,
(h) $\nu(x,t) < 1$,
(i) $\nu(x,t) = 0$ *if and only if* $x = 0$,
(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,
(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x + y, t + s)$,
(l) $\nu(x, .) : (0, \infty) \to [0,1]$ *is continuous,*
(m) $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 1$.
In this case, (μ, ν) *is called an intuitionistic fuzzy norm.*

Example 2.3. *Let* $(X, \|.\|)$ *be a normed space. Denote* $a * b = ab$ *and* $a \diamond b = \min(a + b, 1)$ *for all* $a, b \in [0,1]$ *and let* μ_0 *and* ν_0 *be fuzzy sets on* $X^2 \times (0, \infty)$ *defined as follows:*

$$\mu_0(x,t) = \frac{t}{t + \|x\|}, \text{ and } \nu_0(x,t) = \frac{\|x\|}{t + \|x\|}$$

for all $t \in \mathbb{R}^+$. *Then* $(X, \mu, \nu, *, \diamond)$ *is an intuitionistic fuzzy normed space.*

Definition 2.9. *Let* $(X, \mu, \nu, *, \diamond)$ *be an IFNS. For* $t > 0$, *we define open ball* $B(x, r, t)$ *with center* $x \in X$ *and radius* $0 < r < 1$, *as*

$$\mathcal{B}(x, r, t) = \{y \in X : \mu(x - y, t) > 1 - r, \nu(x - y, t) < r\}.$$

Definition 2.10. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, t) > 1 - \epsilon$ and $\nu(x_k - L, t) < \epsilon$ for all $k \geq k_0$. In this case we write $(\mu, \nu) - \lim x = L$.*

Definition 2.11. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_l, t) < \epsilon$ and $\nu(x_k - x_l, t) < \epsilon$ for all $k, l \geq k_0$.*

3 Kuratowski Statistical and Kuratowski $\mathcal{I}$-Convergence

In this section, we recall some basic properties of Kuratowski convergence. We use the following notation:

$$\begin{aligned} \mathcal{N} &:= \{N \subseteq \mathbb{N} : \mathbb{N} \backslash N \text{ finite}\} \\ &:= \{\text{subsequences of } \mathbb{N} \text{ containing all n beyond some } n_0\} \\ \mathcal{N}^\sharp &:= \{N \subseteq \mathbb{N} : N\text{infinite}\} = \{\text{all subsequences of } \mathbb{N}\}. \end{aligned}$$

We write $\lim\limits_{n\to\infty}$ when $n \to \infty$ as usual in $\mathbb{N}$, but $\lim\limits_{n \in N}$ in this case of convergence of a subsequence designated by an index set N in $\mathcal{N}^\sharp$.

Definition 3.1. *For a sequence (E_n) of closed subsets of X; the outer limit is the set*

$$\begin{aligned} \limsup_{n\to\infty} E_n &:= \{x | \forall \epsilon > 0, \exists N \in \mathcal{N}^\sharp, \forall n \in N : E_n \cap B(x, \epsilon) \neq \phi\} \\ &:= \{x | \exists N \in \mathcal{N}^\sharp, \forall n \in N, \exists x_n \in E_n : \lim_{n \in N} x_n = x\}, \end{aligned}$$

while the inner limit is the set

$$\begin{aligned} \liminf_{n\to\infty} E_n &:= \{x | \forall \epsilon > 0, \exists N \in \mathcal{N}, \forall n \in N : E_n \cap B(x, \epsilon) \neq \phi\} \\ &:= \{x | \exists N \in \mathcal{N}, \forall n \in N, \exists x_n \in E_n : \lim_{n \in N} x_n = x\} \end{aligned}$$

The limit of a sequence (E_n) of closed subsets of X exists if the outer and inner limits sets are equal, i.e,

$$\lim_{n\to\infty} E_n = \liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n.$$

Talo et al. [19] introduced Kuratowski statistical convergence of sequences of closed sets. The statistical outer limit and statistical inner limit of a sequence E_n of closed subsets of X are defined by

$$st - \limsup_{n\to\infty} E_n := \{x | \forall \epsilon > 0, \exists N \in \mathcal{S}^\sharp, \forall n \in N : E_n \cap B(x, \epsilon) \neq \phi\},$$

$$st-\liminf_{n\to\infty} E_n := \{x|\forall\epsilon>0, \exists N\in\mathcal{S}, \forall n\in N : E_n\cap B(x,\epsilon)\neq\phi\},$$

where,

$$\mathcal{S} := \{N\subseteq\mathbb{N} : \delta(N)=1\} \text{ and } \mathcal{S}^{\sharp} := \{N\subseteq\mathbb{N} : \delta(N)\neq 0\}.$$

The statistical limit of a sequence (E_n) exists if its statistical outer and statistical inner limits coincide; ie,

$$st-\lim_{n\to\infty} E_n = st-\liminf_{n\to\infty} E_n = st-\limsup_{n\to\infty} E_n.$$

Further, Talo and Sever [20] introduced Kuratowski $\mathcal{I}$-convergence of sequences of closed sets.

Definition 3.2. *$\mathcal{I}$-outer limit and $\mathcal{I}$-inner limit of a sequence E_n of closed subsets of X are defined by*

$$\mathcal{I}-\limsup_{n\to\infty} E_n := \{x|\forall\epsilon>0, \exists N\in N_I^{\sharp}, \forall n\in N : E_n\cap B(x,\epsilon)\neq\phi\}$$

and

$$\mathcal{I}-\liminf_{n\to\infty} E_n := \{x|\forall\epsilon>0, \exists N\in N_I, \forall n\in N : E_n\cap B(x,\epsilon)\neq\phi\}$$

where $\mathcal{N}_{\mathcal{I}} := \{N\subset\mathbb{N} : \mathbb{N}\backslash N\in\mathcal{I}\} = \mathcal{F}(\mathcal{I})$ and $\mathcal{N}_{\mathcal{I}^{\sharp}} := \{N\subseteq\mathbb{N} : N\notin\mathcal{I}\}$.

The $\mathcal{I}$- limit of a sequence (E_n) exists if its statistical $\mathcal{I}$-outer and $\mathcal{I}$-inner limits coincide. In this situation we say that the sequence of its is Kuratowski $\mathcal{I}$-convergent and we write

$$\mathcal{I}-\lim_{n\to\infty} E_n = \mathcal{I}-\liminf_{n\to\infty} E_n = \mathcal{I}-\limsup_{n\to\infty} E_n.$$

Moreover, it's clear from the inclusion $\mathcal{N}_{\mathcal{I}}\subset\mathcal{N}_{\mathcal{I}}^{\sharp}$ that

$$\mathcal{I}-\liminf_{n\to\infty} E_n \subseteq \mathcal{I}-\limsup_{n\to\infty} E_n$$

so that in fact, $\mathcal{I}-\lim_{n\to\infty} E_n = E$ if and only if

$$\mathcal{I}-\limsup_{n\to\infty} E_n \subseteq A \subseteq \mathcal{I}-\liminf_{n\to\infty} E_n$$

4 Kuratowski IF-I-Convergence

In this section, we introduce Kuratowski IF-I-convergence of double sequences of closed sets. We use the analogous idea employed by Kuratowski [10] and Talo et al. [19] for convergence and statistical convergence of double sequences closed sets. Let us consider

$$N_I := \{N\subseteq\mathbb{N}\times\mathbb{N} : \mathbb{N}\times\mathbb{N}-N\in I\} = \mathcal{F}(I) \text{ and } N_I^{\sharp} := \{N\subseteq\mathbb{N}\times\mathbb{N} : N\notin I\}$$

Firstly, we define the I analogues for outer and inner limits of a double sequence of closed sets.

Definition 4.1. *The $IF-I$-outer limit and $IF-I$-inner limit of a double sequence (A_{ij}) of closed subsets of X are defined as follows:*

$$I_{(\mu,\nu)} \lim_{i,j\to\infty} \sup A_{ij} := \{x|\forall\epsilon > 0, \exists N \in N_I^{\sharp}, \forall(i,j) \in N : A_{ij} \cap \mathcal{B}_x(r,t) \neq \emptyset\},$$

and

$$I_{(\mu,\nu)} \lim_{i,j\to\infty} \inf A_{ij} := \{x|\forall\epsilon > 0, \exists N \in N_I, \forall(i,j) \in N : A_{ij} \cap \mathcal{B}_x(r,t) \neq \emptyset\},$$

The $I_{(\mu,\nu)}$-limit of a sequence (A_{ij}) exists if its $IF-I-$ outer and $IF-I$-inner limits coincide. In this situation, we say that the double sequence of sets is Kuratowski IF-I-convergent and we write

$I_{(\mu,\nu)} \lim_{i,j\to\infty} \inf A_{ij} = I_{(\mu,\nu)} \lim_{i,j\to\infty} \sup A_{ij} = I_{(\mu,\nu)} \lim_{i.j\to\infty} A_{ij}$

Moreover, it is clear from the inclusion $N_I \subset N_I^{\sharp}$ that

$$I_{(\mu,\nu)} \lim_{i,j\to\infty} \inf A_{ij} \subseteq I_{(\mu,\nu)} \lim_{i,j\to\infty} \sup A_{ij}$$

so that in fact, $I_{(\mu,\nu)} - \lim_{i,j\to\infty} A_{ij} = A$ if and only if

$$I_{(\mu,\nu)} \lim_{i,j\to\infty} \sup A_{ij} \subseteq A \subseteq I_{(\mu,\nu)} \lim_{i,j\to\infty} \inf A_{ij}$$

Remark 4.1. *$I_{(\mu,\nu)} - \lim_{i,j\to\infty} A_{ij} = A$ if and only if the following conditions are satisfied:*

(i) for every $x \in A$ and for every $\epsilon > 0$ we have $\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{B}_x(r,t) \cap A_{ij} \neq \emptyset\} \in \mathcal{F}(I)$;

(ii) for every $x \in X\ A$ there exists $\epsilon > 0$ such that $\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{B}_x(r,t) \cap A_{ij} = \emptyset\} \in \mathcal{F}(I)$

We give some examples of ideals and corresponding IF-I-convergence.

(I) Put $I_0 = \{\emptyset\}$. I_0 is minimal ideal in $\mathbb{N}\times\mathbb{N}$. Then for a sequence (A_{ij}) of closed sets we have

$$I_{(\mu,\nu)_0} - \lim_{i,j\to\infty} \inf A_{ij} = \bigcap_{i,j=1}^{\infty} A_{ij} \text{ and } I_{(\mu,\nu)_0} - \lim_{i,j\to\infty} \sup A_{ij} = cl \bigcup_{i,j=1}^{\infty} A_{ij},$$

where cl(A) denotes the closure of the set A in Intuitionistic fuzzy normed space $(X,\mu,\nu,*,\diamond)$. A sequence (A_{ij}) is Kuratowski- $IF-I_0$-convergent if and only if it is constant set.

(II) Take for I the class I_f of all finite subsets of $\mathbb{N}\times\mathbb{N}$. Then I_f is a non-trivial admissible ideal and Kuratowski $IF-I_f$ convergence coincides with the usual Kuratowski I-convergence.

(III) Denote by I_δ the class of all $A \subset \mathbb{N}\times\mathbb{N}$ with $\delta(A) = 0$. Then I_δ is non-trivial admissible ideal and Kuratowski $IF-I_\delta$-convergence coincides with the Kuratowski statistical convergence.

Note that if I is an admissible, then $I_f \subseteq I$. It is clear that

$$\lim_{i,j\to\infty} \inf A_{ij} \subseteq I - \lim_{i,j\to\infty} \inf A_{ij} \subseteq I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf A_{ij} \subseteq I_{(\mu,\nu)} - \lim_{i,j\to\infty} \sup A_{ij}$$
$$\subseteq I - \lim_{i,j\to\infty} \sup A_{ij} \subseteq \lim_{i,j\to\infty} \sup A_{ij}.$$

Hence every Kuratowski convergent sequence is Kuratowski-IF-I-convergent, i.e.,

$$\lim_{i,j\to\infty} A_{ij} = A \text{ implies } I_{(\mu,\nu)} - \lim_{i,j\to\infty} A_{ij} = A.$$

But, the converse of this claim does not hold in general.

Example 4.1. *Let $X = \mathbb{R} \times \mathbb{R}$. We decompose the set $\mathbb{N} \times \mathbb{N}$ into countably many disjoint sets*

$$N_{ij} = \{2^{ij-1}(2s-1) : s \in \mathbb{N}\}, (i,j = 1,2,3,.......).$$

It is obvious that $\mathbb{N} \times \mathbb{N} = \bigcup_{i,j=1}^{\infty} N_{ij}$ and $N_{ij} \cap N_{mn} = \emptyset$ for $(i,j) \neq (m,n)$. Denote by I the class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ such that A intersects only a finite number of N_{ij}. It is easy to see that I is an admissible ideal. Define (A_{ij}) as follows: for $ij \in N_{ij}$ we put

$$A_{ij} = \{x \in \mathbb{R} \times \mathbb{R} : \frac{1}{ij+1} \leq x \leq \frac{1}{ij}\}(i,j = 1,2,3,......).$$

Let $\epsilon > 0, t > 0$. Choose $p \in \times\mathbb{N} \times \mathbb{N}$ such that $\frac{1}{p} < \epsilon$. Then

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : A_{ij} \cap \mathcal{B}_0(r,t) = \emptyset\} \subseteq N_1 \cup N_2 \cup .... \cup N_p.$$

Thus

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : A_{ij} \cap \mathcal{B}_0(r,t) = \emptyset\} \in I \text{ i.e; } \{(i,j) \in \mathbb{N} \times \mathbb{N} : A_{ij} \cap \mathcal{B}_0(r,t) \neq \emptyset\} \in \mathcal{F}(I).$$

So $I - \lim_{i,j\to\infty} A_{ij} = 0$ and hence $I_{(\mu,\nu)} - \lim_{i,j\to\infty} A_{ij} = 0$.

However

$$\lim_{i,j\to\infty} \inf A_{ij} = \emptyset \text{ and } \lim_{i,j\to\infty} \sup A_{ij} = \{x \in \mathbb{R} \times \mathbb{R} : x \leq 1\}.$$

Therefore (A_{ij}) is not Kuratowski convergent.

Theorem 4.1. *Let (A_{ij}) be a sequence of closed subsets of $X(IF - NS)$. Then*

$$I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf A_{ij} = \bigcap_{N_{ij} \in N_I^\sharp} cl \bigcup_{(i,j)\in N_{ij}} A_{ij} \text{ and } I_{(\mu,\nu)} - \lim_{i,j\to\infty} \sup A_{ij} = \bigcap_{N_{ij} \in N_I} cl \bigcup_{(i,j)\in N_{ij}} A_{ij}.$$

Proof. We prove only the first equality because the proof of the second one is similar to the first one. Let $x \in I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf A_{ij}$ be arbitrary and $N_{ij} \in N_I^\sharp$ be arbitrary. For every $\epsilon > 0, t > 0$ there exists $N_{11} \in N_I$ such that for every $(i,j) \in N_{11}$

$$A_{ij} \cap \mathcal{B}_x(r,t) \neq \emptyset.$$

From Lemma 2.2 we have $N_{ij} \cap N_{11} \notin N_I^\sharp$. So there exists $n_0 \in N_{ij} \cap N_{11}$ such that $A_{ij_0} \cap \mathcal{B}_x(r,t) \neq \emptyset$. Therefore,

$$\Big(\bigcup_{ij \in N_{ij}} A_{ij}\Big) \cap \mathcal{B}_x(r,t) \neq \emptyset.$$

This means that $x \in cl \bigcup_{ij \in N_{ij}} A_{ij}$. This holds for any $N_{ij} \in N_I^\sharp$.

Consequently, $x \in \bigcap_{N_{ij} \in N_I^\sharp} cl \bigcup_{ij \in N_{ij}} A_{ij}$.

For the reverse inclusion, suppose that $x \notin I_{(\mu,\nu)} - \lim_{i,j \to \infty} \inf A_{ij}$. Then, there exists $\epsilon > 0, t > 0$ such that

$$N_{ij} = \{(i,j) \in \mathbb{N} \times \mathbb{N} : A_{ij} \cap \mathcal{B}_x(r,t) = \emptyset\} \notin I,$$

i.e; $N_{ij} \in N_I^\sharp$. Thus

$$\Big(\bigcup_{(i,j) \in N_{ij}} A_{ij}\Big) \cap \mathcal{B}_x(r,t) = \emptyset.$$

This means that $x \notin cl \bigcup_{(i,j) \in N_{ij}} A_{ij}$. This completes the proof. ■

Remark 4.2. *As a consequence of Theorem 1, for any given double sequence (A_{ij}) the sets $I_{(\mu,\nu)} - \lim_{i,j \to \infty} \inf A_{ij}$ and $I_{(\mu,\nu)} - \lim_{i,j \to \infty} \sup A_{ij}$ are closed.*

Theorem 4.2. *Let (A_{ij}) be a double sequence of closed subsets of X. Then for every $t > 0$*

$$I_{(\mu,\nu)} - \lim_{i,j \to \infty} \inf A_{ij} = \{x | I_{(\mu,\nu)} - \lim_{i,j \to \infty} \mu(x - A_{ij}, t) = 0 \text{ or } I_{(\mu,\nu)} - \lim_{i,j \to \infty} \nu(x - A_{ij}, t) = 1\},$$

$$I_{(\mu,\nu)} - \lim_{i,j \to \infty} \sup A_{ij} = \{x | I_{(\mu,\nu)} - \lim_{i,j \to \infty} \inf \mu(x - A_{ij}, t) = 0 \text{ or } I_{(\mu,\nu)} - \lim_{i,j \to \infty} \inf \nu(x - A_{ij}, t) = 1\}$$

Proof. For any closed set A we have

$$\mu(x - A, t) \geq \epsilon \text{ or } \nu(x - A, t) \leq 1 - \epsilon \Leftrightarrow A \cap \mathcal{B}_x(r,t) = \emptyset.$$

Suppose that $I_{(\mu,\nu)} - \lim_{i,j \to \infty} \mu(x - A_{ij}, t) = 0$ and $I_{(\mu,\nu)} - \lim_{i,j \to \infty} \nu(x - A_{ij}, t) = 1$. Then for every $\epsilon > 0, t > 0$

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(x - A_{ij}, t) \geq \epsilon \text{ or } \nu(x - A_{ij}, t) \leq 1 - \epsilon\} \in I.$$

Then, for every $\epsilon > 0, t > 0$ we obtain

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : A_{ij} \cap \mathcal{B}_x(r,t) = \emptyset\} \in I.$$

This means that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : A_{ij} \cap \mathcal{B}_x(r,t) \neq \emptyset\} \in \mathcal{F}(I).$$

That is , $x \in I_{(\mu,\nu)} - \lim_{i,j \to \infty} \inf A_{ij}$.

Now, we show the reverse inclusion. Let $x \in I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf A_{ij}$. Then for every $\epsilon > 0, t > 0$ there exists $N_{ij} \in N_I$ such that $A_{ij} \cap \mathcal{B}_x(r,t) \neq \emptyset$ for every $ij \in N_{ij}$. Since

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : A_{ij} \cap \mathcal{B}_x(r,t) = \emptyset\} \subseteq \mathbb{N}\times\mathbb{N}\ N_{ij}$$

we have

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : A_{ij} \cap \mathcal{B}_x(r,t) = \emptyset\} \in I.$$

So, we have

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mu(x - A_{ij}, t) \geq \epsilon \text{ or } \nu(x - A_{ij}, t) \leq 1-\epsilon\} \in I.$$

That is, $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \mu(x - A_{ij}, t) = 0$ and $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \nu(x - A_{ij}, t) = 1$.
Similarly, for any closed set A we have

$$\mu(x - A, t) < \epsilon \text{ or } \nu(x - A, t) > 1-\epsilon \Leftrightarrow A \cap \mathcal{B}_x(r,t) \neq \emptyset. \tag{4.1}$$

Suppose that $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf \mu(x - A_{ij}, t) = 0$ and $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf \nu(x - A_{ij}, t) = 1$. Then for every $\epsilon > 0, t > 0$

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mu(x - A_{ij}, t) < \epsilon \text{ or } \nu(x - A_{ij}, t) > 1-\epsilon\} \notin I$$

By (3.2), for every $\epsilon > 0, t > 0$ we obtain

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : A_{ij} \cap \mathcal{B}_x(r,t) \neq \emptyset\} \notin I.$$

This means that $x \in I_{(\mu,\nu)} - \lim_{i,j\to\infty} \sup A_{ij}$.
Now, we show the reverse inclusion. Let $x \in I_{(\mu,\nu)} - \lim_{i,j\to\infty} \sup A_{ij}$. Then for every $\epsilon > 0, t > 0$

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : A_{ij} \cap \mathcal{B}_x(r,t) \neq \emptyset\} \notin I$$

we have

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : A_{ij} \cap \mathcal{B}_x(r,t) = \emptyset\} \in I.$$

Then, we have
$I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf \mu(x - A_{ij}, t) = 0$ and $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf \nu(x - A_{ij}, t) = 1$.
■

Theorem 4.3. *Let (A_{ij}) be a double sequence of closed subsets of X. Then for every $t > 0$*

$$I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf A_{ij} = \{x | \forall (i,j) \in \mathbb{N}\times\mathbb{N}, \exists y_{ij} \in A_{ij} : I_{(\mu,\nu)} - \lim_{i,j\to\infty} y_{ij} = x\}. \tag{4.2}$$

Proof. Let $x \in I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf A_{ij}$ be arbitrary. By Theorem 4.2, $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \mu(x - A_{ij}, t) = 0$ and $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \nu(x - A_{ij}, t) = 1$.
For every $\epsilon > 0, t > 0$

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mu(x - A_{ij}, t) \geq \frac{\epsilon}{2} \text{ or } \nu(x - A_{ij}, t) \leq 1 - \frac{\epsilon}{2}\} \in I.$$

Since A_{ij} is closed, for $(i,j) \in \mathbb{N}\times\mathbb{N}$, there exists $y_{ij} \in A_{ij}$ such that $\mu(x-y_{ij},t) \leq 2\mu(x-A_{ij},t)$ and $\nu(x-y_{ij},t) \geq 2\nu(x-A_{ij},t)$. Now, we define the sequence
$\{y_{ij}|y_{ij} \in A_{ij}, (i,j) \in \mathbb{N}\times\mathbb{N}\}$.
Then $I_{(\mu,\nu)} - \lim_{i,j\to\infty} y_{ij} = x\}$. On the contrary, assume that x belong to the right hand side set of the equality. Then, there exists
$\{y_{ij}|y_{ij} \in A_{ij}, (i,j) \in \mathbb{N}\times\mathbb{N}\}$ such that $I_{(\mu,\nu)} - \lim_{i,j\to\infty} y_{ij} = x\}$. Then for every $\epsilon > 0, t > 0$

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mu(x-A_{ij},t) \geq \epsilon \text{ or } \nu(x-A_{ij},t) \leq 1-\epsilon\} \in I.$$

The inequalities $\mu(x-y_{ij},t) \geq \mu(x-A_{ij},t)$ and $\nu(x-y_{ij},t) \leq \nu(x-A_{ij},t)$ yields the inclusion

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mu(x-A_{ij},t) \geq \epsilon \text{ or } \nu(x-A_{ij},t) \leq 1-\epsilon\}$$

$$\subseteq \{(i,j) \in \mathbb{N}\times\mathbb{N} : \mu(x-y_{ij},t) \geq \epsilon \text{ or } \nu(x-y_{ij},t) \leq 1-\epsilon\}. \text{ So,}$$

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mu(x-A_{ij},t) \geq \epsilon \text{ or } \nu(x-A_{ij},t) \leq 1-\epsilon\}.$$

This means that $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \mu(x-A_{ij},t) = 0$ and $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \nu(x-A_{ij},t) = 1$. By Theorem 4.2, we have $x \in I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf A_{ij}$.
■

The following result is well known in the theory of Kuratowski double convergence. $x \in \lim_{i,j\to\infty} \inf A_{ij}$ if and only if there exist $N_{ij} \in N = N_{I_f}$ and $x_{ij} \in A_{ij}$ for all $(i,j) \in N_{ij}$ such that $\lim_{(i,j)\in N_{ij}} x_{ij} = x$. For Kuratowski-IF-I-convergence, if I has property (AP), then this fact holds.

Corollary 4.1. *Let I be an admissible ideal. If the ideal I has property (AP) then*

$$I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf A_{ij} = \{x|\exists N_{ij} \in N_I \exists, \forall (i,j) \in \mathbb{N}\times\mathbb{N}, \exists y_{ij} \in A_{ij} : \lim_{i,j\in N_{ij}} y_{ij} = x\}. \tag{4.3}$$

Proof. Suppose that I satisfies condition (AP). Let $x \in I_{(\mu,\nu)} - \lim_{i,j\to\infty} \inf A_{ij}$. Then $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \mu(x-A_{ij},t) = 0$ and $I_{(\mu,\nu)} - \lim_{i,j\to\infty} \nu(x-A_{ij},t) = 1$. By condition (AP) we have $I^*_{(\mu,\nu)} - \lim_{i,j\to\infty} \mu(x-A_{ij},t) = 0$ and $I^*_{(\mu,\nu)} - \lim_{i,j\to\infty} \nu(x-A_{ij},t) = 1$. Then there is a set $M \in \mathcal{F}(I)$ such that
$\lim_{(m,n)\in M} \mu(x-A_{mn},t) = 0$ and $\lim_{(m,n)\in M} \nu(x-A_{mn},t) = 1$.
Since A_{mn} is closed, for $(m,n) \in M$, there exists $y_{mn} \in A_{mn}$ such that

$\mu(x-y_{mn},t) \leq 2\mu(x-A_{mn},t)$ and $\nu(x-y_{mn},t) \geq 2\nu(x-A_{mn},t)$.

Now, define the sequence $\{y_{mn}|y_{mn} \in A_{mn}, (m,n) \in M\}$. Then $\lim_{(m,n)\in M} y_{mn} = x$.

On contrary, assume that x belongs to the right hand set of the equality (3.4). Let us define

$$z_{ij} = \begin{cases} y_{ij}, & \text{if } (i,j) \in \mathbb{N}\times\mathbb{N}, \\ \text{arbitrary element of } A_{ij}, & \text{if } (i,j) \notin \mathbb{N}\times\mathbb{N}. \end{cases}$$

Then $I^*_{(\mu,\nu)} - \lim_{i,j\to\infty} z_{ij} = x$. So $I_{(\mu,\nu)} - \lim_{i,j\to\infty} z_{ij} = x$. By Theorem 4.3, we have $x \in I_{(\mu,\nu)} - \lim_{ij\to\infty} \inf A_{ij}$. ■

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