Numerical Solution of Singularly Perturbed Two- Point Boundary Value Problem using Transformation technique using Quadrature method

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Abstract: We are Fascinated to recount a Green shift in this research paper to resolve a specifically motivated two point limit confidence problem with end limit layer in the stretch [0,1]. Here, we've applied the well-known Greens adjustment to a specific problem and investigated the appropriateness of the mathematical arrangement. Quadrature technique was used during the procedure. Then we used this technique on two straight models with a right end limit layer that were incredibly rough to the particular arrangement. The numerically computed results were compared with the analytical solutions for exactness and to evaluate the error bound. For the assembly of the plan, computationally obtained Findings were discussed, which are in appropriate agreement with the particular arrangement that is available in the literature. Computational results are closely associated with the analytical solutions available in the literature.

Keywords: Singular Perturbation; ordinary Differential Equation; Boundary Layer; Two-Point Boundary Value Problem; Quadrature method, Green Transform.

1. Introduction

In Fluid dynamics, quantum mechanics, ideal control hypothesis, compound responses similarity hypothesis, response propagation cycles, and geophysics, single bother issues are a common occurrence. The two-point limit esteem problem gets a limit or inside layers, i.e. locales of quick shift in the arrangement close to the end Focuses or some inside Focuses with width O(1) as 0 in the uniquely irritated two-point limit esteem issue. Countless speciFic methods have recently been developed to provide detailed mathematical arrangements. (Andreev, V. B. 1996; Axelsson, Nikolova.M, 1988) are good places to look For nuances. -It is present in a large number of these techniques.

(Gear.C.W 1967; Han. H 1990; Il'in. A. M 1969) introduced a not -asymptotic technique, also known as the boundary value approach, for dealing with certain types of singular perturbation problems. Also pointed out at the ingenious and imprecise arrangements of a Few mathematical models. To the best of our knowledge, very few asymptotic solutions were established for boundary value problems (N. Srinivasacharyulu 2008; Vigo –Aguiar.J, S. 2004). In this article, the author discusses two point limit esteem problems with the right end limit layer using the Green shift and comes up with asymptotic and mathematical solutions. There aren't many models on display for the technique's applicability. Perhaps recourse can be developed to solve such kind of singularly perturbed two point boundary value problems.

2 .Greens Transform

We look at the suggested strategy for two – point limit esteem issues with the right – end limit layer of the predefined stretch in this article. To be clear, we're talking about a class structural issues that are particularly bothersome.

$$\varepsilon y'' - F(x)y'(x) - g(x)y(x) = 0, x \in [0,1] \qquad (1)$$

With the defined boundary conditions

$$y(0)=\alpha \text{ and } y(1)=\beta \qquad (2)$$

We assume that F(x),g(x) are unsaid to be adequately consistent differentiable capacities in the predefined interval, where is a little sure boundary (0 1,) and, are established constants. Furthermore, in the range [0,1], the coefficient of y' (x) is negative and non-zero. This presumption strongly suggests that the limit layer would be in the vicinity of x=1. (Right end boundary layer).

Rewrite the Equation (1) as below:

$$-\varepsilon y'' + F(x)y'(x) + g(x)y(x) = 0, x \in [0,1] \qquad (3)$$

Let the new Liouville –Green transforms z,$\varphi(x)$, v(z) be

$$z=\varphi(x)=\frac{\lambda}{\varepsilon}\int F(x)\,dx \quad 0<\lambda \leq 1 \tag{4}$$

In the above integral the limits will be prescribed due to discretization are x_{i-1} to x_{i+1} so that the value of z can be evaluated by Quadrature method. Quadrature method is a process of numerical integration. This process will calculate the unknown definite integral value at each mesh point. While adopting this procedure one has to apply Taylor series approximation and interpolation.

$$\mathbf{y(x_{i+1})-y(x_{i-1})=\int_{x_{i-1}}^{x_{i+1}}}\alpha\ \mathbf{[p(x)\ y(x-\delta)+q(x)\ y(x)\ +\ r(x)]\,dx} \tag{5}$$

Here 0<α< 1 being the known parameter.

By making use of the NEWTON-COTES Formula when n=2 i.e., by applying Simpson's one-third rule We have

$$\mathbf{y(x_{i+1})-y(x_{i-1})=}\alpha\mathbf{\frac{h}{3}[p(x_{i+1})y(x_{i+1}-\delta)+4\,p(x_i)\,y(x_i-\delta)+p(x_{i-1}-\delta)}$$

$$\mathbf{+(p_{i+1}+p_{i-1})\ [y(x_{i+1}-\delta)+y(x_{i-1}-\delta)]+q(x_{i+1})\,y(x_{i+1})+q(x_{i-1})\,y(x_{i-1})+q(x_{i+1})y(x_{i+1})}$$

$$+4q(x_i)y(x_i)+q(x_{i-1})y(x_{i-1})+r(x_{i+1})+4r(x_i)+r(x_{i-1})+r(x_{i+1})+r(x_{i-1})\,] \tag{5 A}$$

Again by utility of Taylor's series expansion we can write with suitable approximation model development without law of generality and getting convergence point of view in the computational solution , We can write it as .

$\mathbf{y(x-\delta)\cong y(x)\text{-}\delta\ dy/dx}$ Here $y^1(x)$ denotes the first derivative.

By approximating $y'(x)$ using linear interpolation method we have

$$y(x_i-\delta)\cong \mathbf{y(x_i)}-\frac{\delta\,\mathbf{[y(x_{i+1})-y(x_{i-1})]}}{2.h}$$

$$=\mathbf{y(x_i)}+\frac{\delta}{\mathbf{2h}}\mathbf{y(x_{i-1})}-\frac{\delta}{\mathbf{2h}}\ \mathbf{y(x_{i+1})} \tag{6}$$

Alike

$$y(x_{i-1}-\delta)\cong(1+\frac{\delta}{h})\ y(x_{i-1})-\frac{\delta}{h}\ y(x_i) \tag{7}$$

$$y(x_{i+1}-\delta)=(1-\frac{\delta}{h})\ y(x_{i+1})+\frac{\delta}{h}\ y(x_i) \tag{8}$$

Hence making use the above equations we can be written the above difference t-ness equation in the simple way of form so , it can be written as using in 5(A) with (6), (7) and (8)

$$y_{i+1}-y_{i-1}\ =\ \frac{h}{3}[\ p_{i+1}[(1-\frac{\delta}{h})\,y_{i+1}\ +\frac{\delta}{h}\ y_i]\ +4p_i[y_i-\frac{\delta}{2h}\ y_{i+1}\ +\frac{\delta}{2h}\ y_{i-1}]+p_{i-1}[(1+\frac{\delta}{h})\,y_{i-1}-\frac{\delta}{h}y_i]$$

$$+(p_{i+1}+p_{i-1})[(1-\frac{\delta}{h})y_{i+1}\ +\frac{\delta}{h}y_i+(1+\frac{\delta}{h})\,y_{i-1}-\frac{\delta}{h}y_i+2q_{i+1}y_{i+1}+2q_{i-1}y_{i-1}+4q_iy_i+2r_{i+1}+4r_i\ +2r_{i-1}\,]$$

$$[-1-\frac{2p_i\,\delta}{3}-\frac{h}{3}p_{i-1}(1+\frac{\delta}{2h})-\frac{h}{3}(p_{i+1}+p_{i-1})(1+\frac{\delta}{h})-\frac{2h}{3}q_{i-1}]\,y_{i-1}+[\frac{\delta\,p_{i-1}}{3}-\frac{\delta}{3}p_{i+1}-\frac{4hp_i}{3}$$

$$-\frac{4hq_i}{3}]\,y_i+[1-\frac{h}{3}p_{i+1}(1-\frac{\delta}{h})+\frac{2p_i\delta}{3}-\frac{h}{3}(p_{i+1}+p_{i-1})(1-\frac{\delta}{h})-\frac{2h}{3}q_{i+1}]\,y_{i+1}$$

$$=\frac{2h}{3}[r_{i+1}+2r_i+r_{i-1}] \tag{9}$$

(9) can be written in the most amicable form as (use of algorithmic approach)

$$A_i\ y_{i-1}+B_i\ y_i+C_i\ y_{i+1}=D_i \tag{10}$$

Here $$A_i = -1 - \frac{2p_i\delta}{3} - \frac{h}{3}p_{i-1}(1+\frac{\delta}{2h}) - \frac{h}{3}(p_{i+1}+p_{i-1})(1+\frac{\delta}{h}) - \frac{2h}{3}q_{i-1} \quad (11)$$

$$B_i = \frac{\delta\,\mathbf{p_{i-1}}}{\mathbf{3}} - \frac{\delta}{3}p_{i+1} - \frac{4\alpha h p_i}{3} - \frac{4hq_i}{3} \quad (12)$$

$$C_i = 1 - \frac{h}{3}p_{i+1}(1-\frac{\delta}{h}) + \frac{2p_i\delta}{3} - \frac{h}{3}(p_{i+1}+p_{i-1})(1-\frac{\delta}{h}) - \frac{2h}{3}q_{i+1} \quad (13)$$

$$D_i = \frac{2h}{3}[r_{i+1} + 2r_i + r_{i-1}] \quad (14)$$

yi = y(xi), pi =p(xi), qi =q(xi), and ri =r(x_i) are the variables in terms of x. Condition (10) produces a set of (N-1) conditions with (N+1) obscuring y0 to y_N. For the questions y0 to y_N, the two given limit conditions (2), as well as these (N-1) conditions, are sufficient. The structure of the Tri-askew (3 diagonal) system (10) can be obtained using the 'Thomas Algorithm,' which is a useful calculation. In this case, it is a diagonally dominant matrix also. So we have

$$y_i = W_i y_{i+1} + T_i \quad (15)$$

Where W_i and T_i correspond to balanced weight functions so that
W(x_i) and T (x_i) are to be determined From (15) we have. Most efficient way is to define

$$y_{i-1} = W_{i-1}\, y_i + T_{i-1} \quad 16)$$

Substituting (16) in (15) we get after suitable re verification we have

$$y_i = \frac{C_i}{B_i - A_i W_{i-1}} y_{i+1} + \frac{A_i T_{i-1} - D_i}{B_i - A_i W_{i-1}} \quad (17)$$

By comparing (15) and (17), we can get

$$W_i = \frac{C_i}{B_i - A_i W_{i-1}} \quad (18)$$

$$T_i = \frac{A_i T_{i-1} - D_i}{B_i - A_i W_{i-1}} \quad (19)$$

We need to know the underlying conditions For W0 and T0 in order to tackle these repeat connections For i=1,2, 3,.......N-1. This should be possible if the limit conditions are taken into account. Convergence criteria is also important in numerical calculation point of view. So we have to consider

$$y_0 = \alpha = W_0 y_1 + T_0 \quad \mathbf{(20)}$$

IF we choose W0=0, T0 will be equal to a known quantity. With these underlying qualities, we sequentially register Wi and Ti For i=1,2,3,...,N-1;in the Forward cycle From (19) and (20), and then acquire yi in the retrogressive relationship From (15) using (2).Repeat the mathematical strategy For different (going wrong contention, meeting the conditions) decisions until the arrangement values not differ significantly From one cycle to the next.For the sake of computational point of view, we use an absolute error criterion, i.e.,

$$\left| y(x)^{m+1} - y(x)^m \right| \le \rho, \mathbf{0 \le x \le 1} \quad \mathbf{(21)}$$

So that underlying embedded algorithmic approach is to define the approximate polynomial as

$$\Phi(x) = \varphi'(x) = \frac{1}{\varepsilon} f(x) \quad (22)$$

$$v(z) = \Phi(x)\, y(x) \quad (23)$$

According to (6) and by term by term differentiation is allowed so that

$$\frac{dy}{dx} = \frac{1}{\Phi(x)}\frac{dv}{dz}z'(x) - \frac{\Phi'(x)}{\Phi^2(x)}v(x) = \frac{\varphi'(x)}{\Phi(x)}\frac{dv}{dz} - \frac{\Phi'(x)}{\Phi^2(x)}v(z)\,, \quad (24)$$

$\frac{d^2y}{dx^2}=\frac{1}{\Phi(x)}\left(\varphi'^2(x)\frac{d^2v}{dz^2}+\left(\Phi''-\frac{2\varphi'(x)\Phi'(x)}{\Phi(x)}\right)\frac{dv}{dz}-\left(\frac{\Phi''(x)}{\Phi(x)}-\frac{2\Phi'^2(x)}{\Phi^2(x)}\right)v\right)$(25)

From (3),(24) and (25) , we obtain

$$-\frac{\varepsilon\varphi'^2}{\Phi}\frac{d^2v}{dz^2}+\left(\frac{2\varepsilon\varphi'\Phi'}{\Phi^2}-\frac{\varepsilon\varphi''(x)}{\Phi(x)}+f(x)\frac{\varphi'(x)}{\Phi(x)}\right)\frac{dv}{dz}+\left(\frac{\varepsilon\Phi''(x)}{\Phi^2(x)}-\frac{2\varepsilon\Phi'^2(x)}{\Phi^3(x)}-f(x)\frac{\Phi'(x)}{\Phi^2}+\frac{g(x)}{\Phi}\right)v(z)=0$$

i.e

$$V''+\frac{1}{\varphi'^2}\left(\varphi''(x)-\frac{2\varphi'\Phi'}{\Phi}-f(x)\frac{\varphi'(x)}{\varepsilon}\right)\frac{dv}{dz}-\frac{1}{\varphi'^2}\left(\frac{\Phi''(x)}{\Phi(x)}-\frac{2\Phi'^2}{\Phi^2}-f(x)\frac{\Phi'(x)}{\varepsilon\Phi(x)}+\frac{g(x)}{\varepsilon}\right)v(z)=0$$

From (21), we have

$$\frac{d^2v}{dz^2}-\left(\varepsilon\frac{f'(x)}{f^2(x)}+1\right)\frac{dv}{dz}-\frac{1}{f^2(x)}\left(\varepsilon^2\frac{f''(x)}{f(x)}-2\varepsilon^2\frac{f'^2(x)}{f^2(x)}-\varepsilon f'(x)+\varepsilon g(x)\right)v(z)=0$$

i.e

$$\frac{d^2v}{dz^2}-\frac{dv}{dz}=\varepsilon\frac{f'(x)}{f^2(x)}\frac{dv}{dz}+\varepsilon\frac{1}{f^2(x)}\left(\varepsilon\frac{f''(x)}{f(x)}-2\varepsilon\frac{f'^2(x)}{f^2(x)}-f'(x)+g(x)\right)v(z)=\varepsilon M(x)\frac{dv}{dz}+\varepsilon N(\varepsilon,x)v(z), \quad (26)$$

Where$M(x)=\frac{f'(x)}{f^2(x)}$,$N(x,\varepsilon)=\frac{1}{f^2(x)}\left(\varepsilon\frac{f''(x)}{f(x)}-2\varepsilon\frac{f'^2(x)}{f^2(x)}-f'(x)+g(x)\right)$

Since ε isasmall parameter ($0<\varepsilon<1$) ,$\varepsilon M(x)$ and $\varepsilon N(x,\varepsilon)$are sufficiently small on [0,1].So, as$\varepsilon\to 0$, theright handsideof Equation (9) becomes ruled out. So we have

So that we can simplify with the above assumptions and qualities of the function selection we have

$$\frac{d^2v}{dz^2}-\frac{dv}{dz}\approx 0. \quad (27)$$

Therefore, the approximate solutions v(z) of(10) are

$$v(z)=C_1+C_2e^z, \quad (28)$$

WhereC1 and C_2 are two arbitrary constants. From (4)-(6), one has the asymptotic solutions oF differential equations

$$y(x)=\frac{v(z)}{\Phi(x)}=\varepsilon\frac{v(z)}{f(x)}\approx\frac{\varepsilon}{f(x)}\left(C_1+C_2e^{\frac{1}{\varepsilon}\int_0^x F(x)dx}\right) \quad (29)$$

whereC_1,C_2 are two arbitrary constants.

3. Applications to Two Point Boundary Value Problems

As an application, we consider the Following second order two-point boundary value problem

$$\varepsilon y''-f(x)y'(x)-g(x)y(x)=0,$$
$$0<\varepsilon<1, 0<x<1, \quad (30)$$

Which is also equivalent as

$$-\varepsilon y''+f(x)y'(x)+g(x)y(x)=0$$
$$y(0)=\alpha, y(1)=\beta,$$

Whereα,β areconstants.

Applying the boundary constants of (1) in(12),we have

$$C_1\frac{\varepsilon}{f(0)}+C_2\frac{\varepsilon}{f(0)}=\alpha,$$

$$C_1\varepsilon\left(\frac{1}{f(1)}\right)+C_2\varepsilon\left(\frac{1}{f(1)}e^{\frac{1}{\varepsilon}\int_0^1 F(x)dx}\right)=\beta$$

One has $C_1=\frac{\varepsilon\alpha\left(\frac{1}{f(1)}e^{\frac{1}{\varepsilon}\int_0^1 F(x)dx}\right)-\frac{\varepsilon\beta}{f(0)}}{|\Delta|}$ (31)

$C_2=\frac{\frac{\varepsilon\beta}{f(0)}-\varepsilon\left(\frac{1}{f(1)}\right)\alpha}{|\Delta|}$ (32)

Where

$\Delta=\frac{\varepsilon^2}{f(0)}\left(\frac{1}{f(1)}\left(e^{\frac{1}{\varepsilon}\int_0^1 F(x)dx}-1\right)\right)$is non –zero.

Then BVP (13) has the Following asymptotic solution:

$y(x) \approx \frac{\varepsilon}{f(x)}\left(C_1 + C_2 e^{\frac{1}{\varepsilon}\int_0^x F(x)dx}\right)$ (33)

WhereC_1, C_2 are given by (2),(3) respectively

Example1. Consider the Following singular perturbation problem

$\varepsilon \frac{d^2 y}{dx^2} - \frac{1.001\, dy}{dx} = 0, x \epsilon [0,1]$ (34)

Withy(0)=1 and y(1)=0. (35)

At x=1, i.e. at the right-end of the simple span, this problem clearly has a limit layer. The precise arrangement is determined by

$y(x) = \left(e^{(x-1)/\varepsilon} - 1\right)/\left(e^{-1/\varepsilon} - 1\right)$

Comparing (1) with (13), we have

F(x)=1, g(x)=0, $\alpha = 1, \beta = 0$

$\Delta = \varepsilon^2\left(e^{\frac{1}{\varepsilon}} - 1\right) \neq 0$

$$C_1 = \frac{\left(e^{\frac{1}{\varepsilon}}\right)}{\varepsilon\left(e^{\frac{1}{\varepsilon}} - 1\right)}$$

$$C_2 = \frac{-1}{\varepsilon\left(e^{\frac{1}{\varepsilon}} - 1\right)}$$

$$y(x) \approx \frac{\left(1 - e^{(x-1)/\varepsilon}\right)}{\left(1 - e^{\frac{-1}{\varepsilon}}\right)} \quad (35)$$

The computational results for = 0.001and 0.0001 are presented separately in Tables 1 and 2. Tables 1 and 2 show our quick response and our more thorough answer for different x estimates.

Table 1. Numerical results of example 1 with , $h = 10^{-3}$, $\varepsilon = 10^{-3}$

X	Numerical solution	Exact Solution
0.000	1.0000000	1.0000000
0.200	1.0000000	1.0000000
0.400	0.9998999	1.0000000
0.600	1.0000000	1.0000000
0.800	1.0000000	1.0000000
0.900	1.0000000	1.0000000
0.920	1.0000000	1.0000000
0.940	0.9899899	1.0000000
0.960	1.0000000	1.0000000
0.979	1.0000000	1.0000000
1.000	1.0000000	1.0000000

Table 2. Numerical results of example 1 with $h = 10^{-4}\varepsilon = 10^{-4}$,

X	Numerical solution	Exact Solution
0.0000	0.9999999	1.0000000
0.2000	1.0000000	1.0000000
0.4000	1.0000000	1.0000000
0.6000	0.9998999	1.0000000
0.8000	1.0000000	1.0000000
0.9000	1.0000000	1.0000000
0.9200	1.0000000	1.0000000
0.9400	1.0000000	1.0000000
0.9600	1.0000000	1.0000000
0.9800	1.0000000	1.0000000
1.0000	1.0000000	1.0000000

Example 2. We take a look at Kevorkian and Cole's variable coefficient singular perturbation problem. [2, p33, Equation (2.3.26) and (2.3.27) with $\alpha = -0.50$

$$\varepsilon . y''(x) - \left(\frac{x}{2} - 1\right) y'(x) - 0.5\, y(x) = 0 \; , x \epsilon [0,1] \tag{36}$$

With y(0)=0 and y(1)=1 (37)

As our "exact" solution can be calculated from analytical methods available in literature to chosen to use uniformly true approximation (which is obtained using the method described by NayFeh[12, p.148, Equation (4.2.32)];

y(x) $= \frac{1}{(2-x)} - \frac{1}{2} e^{-(x-x^2/4)/\varepsilon}$

The numerical results are given in Tables 3 and 4for$\varepsilon = 10^{-3}$ and10^{-4} for better understanding do the comparative principle i.e

Comparing (4) with (13), we have

F(x) = $\left(\frac{x}{2} - 1\right)$, g(x) = $\frac{1}{2}$,$\alpha = 0, \beta = 1$,$\Delta = 2\varepsilon^2 \left(e^{\frac{-3}{4\varepsilon}} - 1\right) \neq 0,$

$C_1 = \frac{\varepsilon}{\Delta}$, $C_2 = \frac{-\varepsilon}{\Delta}$

$$y(x) \approx \frac{1}{(2-\square)} \frac{\left(1 - \square^{\frac{(\square^2 - 4\square)}{4\square}}\right)}{\left(1 - \square^{\frac{-3}{4\square}}\right)} \tag{38}$$

The computational results are presented in **Tables 3** and **4**for$\varepsilon = 10^{-3}$and10^{-4}, respectively. **Figures 3** and **4** show our solution and exact solution for various deviating and different values of x.

x	Numerical solution	Exact Solution
0.000	0.0000000	0.0000000
0.200	0.5555545	0.5555556
0.400	0.6250000	0.6250000
0.600	0.7142857	0.7142857
0.800	0.8333333	0.8333333
0.900	0.9090909	0.9090909
0.920	0.9259259	0.9259259
0.940	0.9433962	0.9433962
0.960	0.9615384	0.9615384
0.980	0.9803922	0.98039922
1.000	1.0000000	1.0000000

Table 3. Numerical results oF example 2 with $\varepsilon = 10^{-3}, h = 10^{-3}$

X	Numerical Solution	Exact Solution
0.0000	0.0000000	0.0000000
0.2000	0.5555555	0.5555556
0.4000	0.6250000	0.6250000
0.6000	0.7142857	0.7142857
0.8000	0.8333333	0.8333333
0.9000	0.9090909	0.9090909
0.9200	0.9259259	0.9259259
0.9400	0.9433962	0.9433962
0.9600	0.9615394	0.9615384
0.9800	0.9803922	0.98039922
1.0000	1.0000000	1.0000000

Table 4. Numerical results oF example 3.2 with $\varepsilon = 10^{-34}, h = 10^{-4}$

4. Observations and Conclusion

In every numerical step by step process we have to concentrate and focus on the solution patterns such a way that the computationally obtained results must satisfies the selected mathematical model in the perceptional point of view to get accuracy and consistency and stability in the solution at each mesh point. For that more care has been taken to get a better approximated solution into consideration while compilation.

The authors of this research paper were interested in the mathematical outcomes (arrangement) of independently frustrated two point limit esteem issues with the right end limit sheet. They assume that F(x) has an overall span of [0,1] in this case i.e., in the general range [0,1], the power F(x) has the same symbol. On a computer, our approach works well. But this method is not valid if the equation is changed into the form.

$-\varepsilon y''(x) + F(x)y'(x) + g(x)y(x) = h(x) \neq 0$ i.e in non- homogeneous linear or non-linear form. As we can try in our next attempt to address this kind of problems also.

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